

On Hopf algebras and the elimination theorem for free Lie algebras

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Abstract

The elimination theorem for free Lie algebras, a general principle which describes the structure of a free Lie algebra in terms of free Lie subalgebras, has been recently used by E. Jurisich to prove that R. Borchers' "Monster Lie algebra" has certain large free Lie subalgebras, illuminating part of Borchers' proof that the moonshine module vertex operator algebra obeys the Conway-Norton conjectures. In the present expository note, we explain how the elimination theorem has a very simple and natural generalization to, and formulation in terms of, Hopf algebras. This fact already follows from general results contained in unpublished 1972 work, unknown to us when we wrote this note, of R. Block and P. Leroux.

1 Introduction

The elimination theorem for free Lie algebras (see [10] and [3]) describes how to "eliminate" generators; we state and discuss this theorem in detail below. It is of interest to us because it is an important step in E. Jurisich's proof of her theorem that R. Borchers' "Monster Lie algebra" [2] is "almost free": This Lie algebra is the direct sum of the 4-dimensional Lie algebra $\mathfrak{gl}(2)$ and two $\mathfrak{gl}(2)$ -submodules, each of which is a free Lie algebra over a specified infinite-dimensional $\mathfrak{gl}(2)$ -module constructed naturally from the moonshine module ([5], [6]) for the Fischer-Griess Monster; see [7], [8]. This theorem has been used in [7], [8] and further, in [9], to illuminate and simplify Borchers' proof [2] that the moonshine module vertex operator algebra satisfies the conditions of the Conway-Norton monstrous-moonshine conjectures [4]: The McKay-Thompson series for the the action of the Monster on this structure are modular functions which agree with the modular functions conjectured in [4] to be associated with a graded Monster-module. In fact, Jurisich's theorem, including its use

of the elimination theorem, applies to the more general class of generalized Kac-Moody algebras, also called Borcherds algebras, which have no distinct orthogonal imaginary simple roots; again see [7], [8], and also [9] for further results.

The purpose of this note is to explain how the elimination theorem for free Lie algebras has a very natural and simple generalization to, and formulation in terms of, Hopf algebras (and in particular, quantum groups).

After completing this note we learned from Richard Block that our results follow from a still more general theorem contained in unpublished work [1] of R. Block and P. Leroux, which describes certain adjoint functors (cf. Remark 3.1 below). The present note, then, should be viewed as an exposition showing that the categorical considerations of Block and Leroux have concrete implications for the study of the Monster Lie algebra and monstrous moonshine.

Let us write $F(S)$ for the free Lie algebra over a given set S and $T(S)$ for the free associative algebra (the tensor algebra) over S . The elimination theorem states (see [3]) that the free Lie algebra over the disjoint union of two sets R and S is naturally isomorphic to the semidirect product of the free Lie algebra $F(S)$ with an ideal consisting of another free Lie algebra. This ideal is the ideal of $F(R \cup S)$ generated by R , and it is the free Lie algebra $F(T(S) \cdot R)$, where the dot denotes the natural adjoint action of $T(S)$, which is the universal enveloping algebra of $F(S)$, in $F(R \cup S)$:

$$F(R \cup S) = F(T(S) \cdot R) \rtimes F(S);$$

moreover, this $F(S)$ -module $T(S) \cdot R$ is in fact the free $F(S)$ -module generated by the set R . (“Module” means “left module” unless “right module” is specified.)

The main result of this note is a general structure theorem (which, we repeat, follows from the results in the unpublished work [1]) for the algebra freely generated by a given Hopf algebra and a given vector space, in a sense that we explain below. (“Algebra” means “associative algebra with unit.”) The theorem expresses this freely generated algebra as the smash product of the given Hopf algebra with the tensor algebra over the module for the Hopf algebra freely generated by the given vector space. (We also point out that this freely generated algebra is a Hopf algebra itself in a natural way.) From the special case of this theorem when the given Hopf algebra is the universal enveloping algebra of some Lie algebra, we obtain as an immediate corollary a structure theorem expressing the Lie algebra freely generated by a given Lie algebra and a given vector space as the semidirect product of the given Lie algebra and the free Lie algebra over the module for the Lie algebra freely generated by the given vector space. Finally, from the further special case when this given Lie algebra is free, we recover the elimination theorem as an immediate corollary. (Actually, what we recover is the statement of the elimination theorem using free Lie algebras over vector spaces rather than free Lie algebras over sets; the distinction between the two equivalent formulations of the elimination theorem is simply a

choice of basis for the underlying vector spaces, and such a choice is not necessary or natural for our purposes.) It is in this sense that we are “explaining” the elimination theorem as a very special manifestation of a general Hopf algebra principle.

The first of these two corollaries (the corollary involving a general Lie algebra) illuminates Jurisich’s use of the elimination theorem in [7] and [8], and in fact makes her theorem exhibiting free Lie subalgebras of generalized Kac-Moody algebras, including the Monster Lie algebra, quite transparent. For the case of the Monster Lie algebra, we take the Lie algebra in our setting to be the Lie subalgebra $\mathfrak{gl}(2)$ of the Monster Lie algebra, and the vector space in our setting to be the moonshine module modulo its (one-dimensional) lowest weight space; then from the definition [2], [7], [8] of the Monster Lie algebra in terms of generators and relations, Jurisich’s result follows immediately. This argument in fact really amounts to her argument in [8], with the difference that the Lie subalgebra $\mathfrak{gl}(2)$ is viewed as already given, rather than being presented in terms of generators and relations. Our main point is that what is really at work here is a very general, simple and natural Hopf-algebra principle. Moreover, our emphasis on tensor algebras (which are the universal enveloping algebras of free Lie algebras) illuminates the structure, presented in Section 4 of [9], of certain standard modules for the Monster Lie algebra as generalized Verma modules—as tensor algebras over certain modules for the Monster (see especially Theorem 4.5 of [9]).

This work is a continuation of some material in a talk of J. Lepowsky’s at the June, 1994 Joint Summer Research Conference on Moonshine, the Monster and Related Topics at Mount Holyoke College, at which [9] was presented. J. L. wishes to thank the organizers, Chongying Dong, Geoff Mason and John McKay, for a stimulating conference. The authors are grateful to Elizabeth Jurisich, Wanglai Li and Siu-Hung Ng, and, as noted above, Richard Block, for helpful comments related to this work. Both authors are partially supported by NSF grant DMS-9401851.

2 The results

In this section, we formulate the main theorem and deduce its consequences. We prove the theorem in the next section.

Throughout this note, we work over a field \mathbb{F} . Let H be a Hopf algebra, equipped with multiplication $M : H \otimes H \rightarrow H$, unit $u : \mathbb{F} \rightarrow H$, comultiplication (diagonal map) $\Delta : H \rightarrow H \otimes H$, counit $\epsilon : H \rightarrow \mathbb{F}$ and antipode $S : H \rightarrow H$, satisfying the usual axioms (cf. [11]). That is, H is an algebra with multiplication M (typically expressed as usual by juxtaposition of elements) and unit element $u(1)$ ($1 \in \mathbb{F}$), which we also write as 1_H , and is also a coalgebra such that the coalgebra operations Δ and ϵ are algebra homomorphisms (i.e., H is a bialgebra), and H is equipped with a linear endomorphism S such that

for $h \in H$,

$$\sum_i h_{1i} S(h_{2i}) = u\epsilon(h) = \epsilon(h)1_H = \sum_i S(h_{1i})h_{2i}. \quad (1)$$

Here and throughout this note, we use the following conventions for expressing the comultiplication (in any coalgebra) and its iterates: For $h \in H$, we write

$$\Delta(h) = \sum_i h_{1i} \otimes h_{2i}, \quad (2)$$

where h_{1i} and h_{2i} are suitable elements of H ; we do not specify any index set in the summation. We similarly write

$$(\Delta \otimes 1)\Delta(h) = \sum_i h_{1i} \otimes h_{2i} \otimes h_{3i}, \quad (3)$$

where the index set and the elements h_{ji} in (2) and (3) are unrelated even though the notations are similar. (Note that by the coassociativity, the expression in (3) also equals $(1 \otimes \Delta)\Delta(h)$.) More generally, for $n > 0$ we write the result of applying the n -fold diagonal map to h as $\sum_i h_{1i} \otimes h_{2i} \otimes \cdots \otimes h_{n+1,i}$.

Note that the defining condition (1) for the antipode can also be written:

$$M \circ (1 \otimes S) \circ \Delta = u\epsilon = M \circ (S \otimes 1) \circ \Delta \quad (4)$$

as maps from H to H .

The antipode S is an algebra and coalgebra antimorphism (cf. [11]).

Our main object of interest is the algebra freely generated by H and a given vector space. First we define the algebra freely generated by a given algebra and a given vector space.

Given an algebra A and a vector space V , let $\mathcal{A}(A, V)$ denote the algebra freely generated by the algebra A and the vector space V , in the natural sense specified by the following universal property:

There is a given algebra homomorphism $A \rightarrow \mathcal{A}(A, V)$ and a given vector space homomorphism $V \rightarrow \mathcal{A}(A, V)$ such that for any algebra \mathcal{B} and any algebra homomorphism $A \rightarrow \mathcal{B}$ and vector space homomorphism $V \rightarrow \mathcal{B}$, there is a unique algebra homomorphism $\mathcal{A}(A, V) \rightarrow \mathcal{B}$ such that the two obvious diagrams commute. Such a structure is of course unique up to unique isomorphism if it exists.

We shall now exhibit such a structure, confirming the existence. In the definition, we have not assumed that either A or V is embedded in $\mathcal{A}(A, V)$, but the construction which follows shows that both of these are in fact embedded in $\mathcal{A}(A, V)$.

Set

$$\mathcal{A}_n = (A \otimes V)^{\otimes n} \otimes A \text{ for } n \in \mathbb{N} \quad (5)$$

and

$$\mathcal{A}(A, V) = \coprod_{n \in \mathbb{N}} \mathcal{A}_n. \quad (6)$$

Then $\mathcal{A}(A, V)$ is an \mathbb{N} -graded algebra in a natural way; the product of an element of \mathcal{A}_m with an element of \mathcal{A}_n is obtained by juxtaposition followed multiplication of the adjacent A -factors. This structure clearly satisfies the universal property. Note that $\mathcal{A}(A, V)$ contains A as a subalgebra (in fact, $A = \mathcal{A}_0$) and V as a vector subspace ($V \subset \mathcal{A}_1$). It also contains the tensor algebras $T(A \otimes V)$ and $T(V \otimes A)$ as natural subalgebras, but $\mathcal{A}(A, V)$ is not the tensor product of either of these tensor algebras with the algebra A .

We shall take A to be the Hopf algebra H , and consider the algebra $\mathcal{A}(H, V)$, which contains H as a subalgebra.

Whenever the Hopf algebra H is a subalgebra of an algebra C , we have the following natural “adjoint” action of H on C : For $h \in H$ and $c \in C$,

$$h \cdot c = \sum_i h_{1i} c S(h_{2i}). \quad (7)$$

It is easy to check from the definitions and the fact that S is an algebra anti-morphism that this action makes the algebra C an H -module algebra, where H is now viewed as a bialgebra rather than a Hopf algebra. This means (cf. [11]) that the linear action $H \otimes C \rightarrow C$ makes C an H -module (where H is viewed as an algebra and C as a vector space) and measures C to C (where H is viewed as a coalgebra and C as an algebra). This last condition means that for $h \in H$ and $c_1, c_2 \in C$,

$$h \cdot c_1 c_2 = \sum_i (h_{1i} \cdot c_1)(h_{2i} \cdot c_2) \quad (8)$$

and

$$h \cdot 1_C = u\epsilon(h) = \epsilon(h)1_C. \quad (9)$$

Note that while the definition of “ H -module algebra” requires only that H be a bialgebra and not a Hopf algebra, the definition of the adjoint action (7) of H on C uses the antipode. Note also (the case $C = H$) that under the action (7), any Hopf algebra is a module algebra for itself. From the definitions we see that the product in C and the action (7) are related as follows: For $h \in H$ and $c \in C$,

$$hc = \sum_i (h_{1i} \cdot c) h_{2i}. \quad (10)$$

We also need the notion of smash product algebra $A \# B$ of an algebra A which is a B -module algebra, B a bialgebra, with the bialgebra B (cf. [11]): As a vector space, $A \# B$ is $A \otimes B$, and multiplication is defined by:

$$(a_1 \otimes b^1)(a_2 \otimes b^2) = \sum_i a_1(b_{1i}^1 \cdot a_2) \otimes b_{2i}^1 b^2 \quad (11)$$

for $a_j \in A$, $b^j \in B$. Suppose that C is an algebra containing subalgebras A and H ; that H is a Hopf algebra; that A is stable under the action (7); and that C is linearly the tensor product $A \otimes H$ in the sense that the natural linear map

$A \otimes H \rightarrow C$ induced by multiplication in C is a linear isomorphism. Then it is clear from (10) that (11) holds (for $B = H$) and hence that

$$C = A \# H. \quad (12)$$

For a vector space V , we shall write $T(V)$ for the tensor algebra over V . (Recall that in the Introduction, we also used the notation $T(S)$ for the free associative algebra over a set S ; then $T(V)$ is naturally isomorphic to $T(S)$ for any given basis S of V .)

We are now ready to state the main result. The proof is given in the next section. Note that statements (1), (2) and (3) in the Theorem include three different “freeness” assertions.

Theorem 2.1 *Let H be a Hopf algebra and V a vector space, and consider the algebra $\mathcal{A} = \mathcal{A}(H, V)$ freely generated by H and V and the canonical adjoint action (7) of H on \mathcal{A} , making \mathcal{A} an H -module algebra. Then:*

1. *The H -submodule $H \cdot V$ of \mathcal{A} generated by V is naturally isomorphic to the free H -module over V .*
2. *The subalgebra of \mathcal{A} generated by $H \cdot V$ is naturally isomorphic to the tensor algebra $T(H \cdot V)$.*
3. *As a vector space, the algebra \mathcal{A} is naturally isomorphic to the tensor product*

$$\mathcal{A} = T(H \cdot V) \otimes H$$

under multiplication in \mathcal{A} ; $T(H \cdot V)$ is an H -module subalgebra of \mathcal{A} ; and with this structure, \mathcal{A} is the smash product of $T(H \cdot V)$ and H :

$$\mathcal{A} = T(H \cdot V) \# H.$$

Here we observe several consequences. We describe first the special case in which H is the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra \mathfrak{g} and then the further special case in which \mathfrak{g} is the free Lie algebra over a vector space. For a vector space W , we shall write $F(W)$ for the free Lie algebra over W , so that $F(W)$ is naturally isomorphic to the free Lie algebra $F(S)$ for any given basis S of W . (As with the notation $T(\cdot)$, we are using the notation $F(\cdot)$ for both sets and vector spaces.)

Corollary 2.2 *Let \mathfrak{g} be a Lie algebra and V a vector space. Let \mathcal{A} be the algebra freely generated by \mathfrak{g} and V (defined by the obvious universal property), or equivalently, the algebra freely generated by $U(\mathfrak{g})$ and V , so that $\mathfrak{g} \subset U(\mathfrak{g}) \subset \mathcal{A}$. Let \mathfrak{a} be the Lie subalgebra of \mathcal{A} generated by \mathfrak{g} and V . Then \mathfrak{a} is naturally isomorphic to the Lie algebra freely generated by \mathfrak{g} and V (in the obvious sense). Moreover, \mathcal{A} is naturally isomorphic to the universal enveloping algebra of \mathfrak{a} . Consider the adjoint action of \mathfrak{g} on \mathcal{A} ; then the associated canonical action of $U(\mathfrak{g})$ on \mathcal{A} is the adjoint action (7). We have:*

1. The \mathfrak{g} -module $U(\mathfrak{g}) \cdot V \subset \mathfrak{a} \subset \mathcal{A}$ generated by V is naturally isomorphic to the free \mathfrak{g} -module (i.e., free $U(\mathfrak{g})$ -module) over V .
2. The (associative) subalgebra of \mathcal{A} generated by $U(\mathfrak{g}) \cdot V$ is naturally isomorphic to the tensor algebra $T(U(\mathfrak{g}) \cdot V)$ and in particular, the Lie subalgebra of \mathcal{A} , and of \mathfrak{a} , generated by $U(\mathfrak{g}) \cdot V$ is naturally isomorphic to the free Lie algebra $F(U(\mathfrak{g}) \cdot V)$.
3. The algebra \mathcal{A} is the smash product

$$\mathcal{A} = T(U(\mathfrak{g}) \cdot V) \# U(\mathfrak{g})$$

and in particular,

$$\mathfrak{a} = F(U(\mathfrak{g}) \cdot V) \rtimes \mathfrak{g};$$

$F(U(\mathfrak{g}) \cdot V)$ is the ideal of \mathfrak{a} generated by V .

Proof: All of this is clear; to verify the second part of (3), we note that $F(U(\mathfrak{g}) \cdot V) \oplus \mathfrak{g} \subset \mathfrak{a}$, and the adjoint action of \mathfrak{g} on \mathcal{A} preserves $F(U(\mathfrak{g}) \cdot V)$. Thus $F(U(\mathfrak{g}) \cdot V) \oplus \mathfrak{g}$ is a Lie subalgebra of \mathfrak{a} containing \mathfrak{g} and V , giving us the semidirect product. \square

Remark 2.3 *The Poincaré-Birkhoff-Witt theorem has been used here in two places: It is used to show that $\mathfrak{g} \subset U(\mathfrak{g})$. It is also used to show that \mathfrak{a} is naturally isomorphic to the Lie algebra freely generated by \mathfrak{g} and V ; the proof of the universal property for \mathfrak{a} —that for any Lie algebra \mathfrak{b} and any Lie algebra map $\mathfrak{g} \rightarrow \mathfrak{b}$ and any vector space map $V \rightarrow \mathfrak{b}$, there exists exactly one Lie algebra map $\mathfrak{a} \rightarrow \mathfrak{b}$ making the two obvious diagrams commute—uses the fact that $\mathfrak{b} \subset U(\mathfrak{b})$. Note that this latter use of the Poincaré-Birkhoff-Witt theorem is a generalization of the similar use of this theorem (in what amounts to the case $\mathfrak{g} = 0$) in proving that the Lie subalgebra of $T(V)$ generated by V is the free Lie algebra over V and hence that $T(V) = U(F(V))$.*

We restate the parts of this corollary involving only the Lie algebra structure:

Corollary 2.4 *Let \mathfrak{g} be a Lie algebra and V a vector space, and let \mathfrak{a} be the Lie algebra freely generated by \mathfrak{g} and V , so that $\mathfrak{g} \subset \mathfrak{a}$ and $V \subset \mathfrak{a}$. Then:*

1. The \mathfrak{g} -module $U(\mathfrak{g}) \cdot V \subset \mathfrak{a}$ generated by V is naturally isomorphic to the free \mathfrak{g} -module over V .
2. The Lie subalgebra of \mathfrak{a} generated by $U(\mathfrak{g}) \cdot V$ is naturally isomorphic to the free Lie algebra $F(U(\mathfrak{g}) \cdot V)$.
3. We have the semidirect product decomposition

$$\mathfrak{a} = F(U(\mathfrak{g}) \cdot V) \rtimes \mathfrak{g},$$

and $F(U(\mathfrak{g}) \cdot V)$ is the ideal of \mathfrak{a} generated by V . \square

Now we consider the further special case in which $\mathfrak{g} = F(W)$, the free Lie algebra over a given vector space W . We have immediately:

Corollary 2.5 *Let V and W be vector spaces, and consider $F(W), F(V \oplus W) \subset T(V \oplus W)$. Then:*

1. *The $F(W)$ -module*

$$U(F(W)) \cdot V = T(W) \cdot V \subset F(V \oplus W)$$

generated by V is naturally isomorphic to the free $F(W)$ -module (i.e., free $T(W)$ -module) over V .

2. *The Lie subalgebra of $F(V \oplus W)$ generated by $T(W) \cdot V$ is naturally isomorphic to the free Lie algebra $F(T(W) \cdot V)$ and the associative subalgebra of $T(V \oplus W)$ generated by $T(W) \cdot V$ is naturally isomorphic to the tensor algebra $T(T(W) \cdot V)$.*

3. *We have the semidirect product decomposition*

$$F(V \oplus W) = F(T(W) \cdot V) \rtimes F(W),$$

$F(T(W) \cdot V)$ is the ideal of $F(V \oplus W)$ generated by V , and

$$T(V \oplus W) = T(T(W) \cdot V) \# T(W). \quad \square$$

Remark 2.6 *The elimination theorem for free Lie algebras over sets rather than over vector spaces, as recalled in the Introduction, is simply the obvious restatement of the last corollary for the free Lie algebra $F(R \cup S)$ over the disjoint union of sets R and S in place of the free Lie algebra $F(V \oplus W)$; we take R and S to be bases of V and W , respectively. Also, $T(V \oplus W)$ is replaced by $T(R \cup S)$.*

Remark 2.7 *Given H and V as in Theorem 2.1, we observe that the algebra $\mathcal{A} = \mathcal{A}(H, V)$ freely generated by H and V is naturally a Hopf algebra. In fact, define $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ to be the unique algebra map which agrees with $\Delta : H \rightarrow H \otimes H$ on H and with the linear map $V \rightarrow V \otimes V$ taking $v \in V$ to $v \otimes 1 + 1 \otimes v$ on V . Define $\epsilon : \mathcal{A} \rightarrow \mathbb{F}$ to be the unique algebra map which agrees with $\epsilon : H \rightarrow \mathbb{F}$ and with the zero map on V . Then \mathcal{A} is clearly a bialgebra since H and V generate \mathcal{A} . Define $S : \mathcal{A} \rightarrow \mathcal{A}$ to be the unique algebra antimorphism which agrees with the given antipode on H and with the map -1 on V . Then S is an antipode, since (4) holds on H and on V , and since the set of elements of a given bialgebra on which (4) holds, where S is a given algebra antimorphism, is closed under multiplication (cf. [11], p. 73).*

3 Proof of Theorem 2.1

Now we prove Theorem 2.1.

Consider the free H -module $H \otimes V$ generated by V (viewing H as an algebra) and define the linear map

$$\begin{aligned} i : H \otimes V &\rightarrow H \otimes V \otimes H \\ h \otimes v &\mapsto \sum_i h_{1i} \otimes v \otimes S(h_{2i}) \end{aligned} \quad (13)$$

($h \in H, v \in V$); that is,

$$i = (1 \otimes \mathcal{T}) \circ (1 \otimes S \otimes 1) \circ (\Delta \otimes 1), \quad (14)$$

where \mathcal{T} is the twist map:

$$\begin{aligned} \mathcal{T} : K \otimes L &\rightarrow L \otimes K \\ k \otimes l &\mapsto l \otimes k \end{aligned} \quad (15)$$

($k \in K, l \in L$) for arbitrary vector spaces K and L . (There should be no confusion between the two uses of the notation “ i .”) Then i is clearly an H -module map, where $H \otimes V \otimes H$ is understood as the tensor product of the free H -module $H \otimes V$ and the H -module H equipped with the (left) action given by: $h \cdot k = kS(h)$ for $h, k \in H$. The image $i(H \otimes V)$ of i is the H -submodule of $H \otimes V \otimes H$ generated by V with respect to this action: $i(H \otimes V) = H \cdot V$. Furthermore, the H -module map

$$i : H \otimes V \rightarrow H \cdot V \quad (16)$$

is an isomorphism (i.e., is an injection), since the linear map

$$1 \otimes 1 \otimes \epsilon : H \otimes V \otimes H \rightarrow H \otimes V$$

is a left inverse of i . In particular, $H \cdot V$ is a copy of the free H -module generated by V . Recalling (5), (6) and writing $\mathcal{A} = \mathcal{A}(H, V)$ as in the statement of Theorem 2.1, we observe that $H \cdot V \subset \mathcal{A}_1 \subset \mathcal{A}$, and that the first part of Theorem 2.1 is verified.

Now consider the linear map

$$\begin{aligned} i_1 : H \otimes V \otimes H &\rightarrow H \otimes V \otimes H \\ h^1 \otimes v \otimes h^2 &\mapsto \sum_i h_{1i}^1 \otimes v \otimes S(h_{2i}^1) h^2 \end{aligned} \quad (17)$$

($h^1, h^2 \in H, v \in V$), which we may express as the map

$$i_1 : \mathcal{A}_1 \rightarrow \mathcal{A}_1 \quad (18)$$

given by the composition of $i \otimes 1 : \mathcal{A}_1 \rightarrow \mathcal{A}_1 \otimes H$ with the multiplication map in \mathcal{A} . We may also write:

$$i_1 = (1 \otimes \mathcal{T}) \circ (1 \otimes M \otimes 1) \circ (1 \otimes S \otimes 1 \otimes 1) \circ (\Delta \otimes 1 \otimes 1) \circ (1 \otimes \mathcal{T}). \quad (19)$$

We now show that i_1 is a linear automorphism, and in fact that the map

$$\begin{aligned} j_1 : H \otimes V \otimes H &\rightarrow H \otimes V \otimes H \\ h^1 \otimes v \otimes h^2 &\mapsto \sum_i h_{1i}^1 \otimes v \otimes h_{2i}^1 h^2 \end{aligned} \quad (20)$$

is a left and right inverse of i_1 : For variety, in this argument we use the symbolism (4) and (19). Since

$$j_1 = (1 \otimes \mathcal{T}) \circ (1 \otimes M \otimes 1) \circ (\Delta \otimes 1 \otimes 1) \circ (1 \otimes \mathcal{T}), \quad (21)$$

we have

$$\begin{aligned} j_1 \circ i_1 &= (1 \otimes \mathcal{T})(1 \otimes M \otimes 1)(\Delta \otimes 1 \otimes 1)(1 \otimes M \otimes 1) \cdot \\ &\quad \cdot (1 \otimes S \otimes 1 \otimes 1)(\Delta \otimes 1 \otimes 1)(1 \otimes \mathcal{T}) \\ &= (1 \otimes \mathcal{T})(1 \otimes M \otimes 1)(1 \otimes 1 \otimes M \otimes 1) \cdot \\ &\quad \cdot (\Delta \otimes 1 \otimes 1 \otimes 1)(1 \otimes S \otimes 1 \otimes 1)(\Delta \otimes 1 \otimes 1)(1 \otimes \mathcal{T}) \\ &= (1 \otimes \mathcal{T})(1 \otimes M \otimes 1)(1 \otimes 1 \otimes M \otimes 1)(1 \otimes 1 \otimes S \otimes 1 \otimes 1) \cdot \\ &\quad \cdot (\Delta \otimes 1 \otimes 1 \otimes 1)(\Delta \otimes 1 \otimes 1)(1 \otimes \mathcal{T}) \\ &= (1 \otimes \mathcal{T})(1 \otimes M \otimes 1)(1 \otimes M \otimes 1 \otimes 1)(1 \otimes 1 \otimes S \otimes 1 \otimes 1) \cdot \\ &\quad \cdot (1 \otimes \Delta \otimes 1 \otimes 1)(\Delta \otimes 1 \otimes 1)(1 \otimes \mathcal{T}) \\ &= (1 \otimes \mathcal{T})(1 \otimes M \otimes 1)(1 \otimes u\epsilon \otimes 1)(\Delta \otimes 1 \otimes 1)(1 \otimes \mathcal{T}) \\ &= (1 \otimes \mathcal{T})(1 \otimes \mathcal{T}) = 1, \end{aligned}$$

where we have used here essentially all the defining properties of a Hopf algebra: associativity, coassociativity, the definition (4) of the antipode, and the unit and counit properties. The fact the $i_1 \circ j_1 = 1$ is verified similarly. (The two parts of (4) are used in the two different arguments.)

For $n \geq 1$, define

$$i_n, j_n : \mathcal{A}_n \rightarrow \mathcal{A}_n \quad (22)$$

by:

$$i_n = (i_1 \otimes 1 \otimes \cdots \otimes 1) \circ \cdots \circ (1 \otimes \cdots \otimes 1 \otimes i_1 \otimes 1 \otimes 1) \circ (1 \otimes \cdots \otimes 1 \otimes i_1), \quad (23)$$

$$j_n = (1 \otimes \cdots \otimes 1 \otimes j_1) \circ (1 \otimes \cdots \otimes 1 \otimes j_1 \otimes 1 \otimes 1) \circ \cdots \circ (j_1 \otimes 1 \otimes \cdots \otimes 1). \quad (24)$$

Using the facts that $j_1 \circ i_1 = 1$, $i_1 \circ j_1 = 1$, we obtain:

$$j_n \circ i_n = 1 \quad (25)$$

$$i_n \circ j_n = 1. \quad (26)$$

Note that we may write $i_2 : \mathcal{A}_2 \rightarrow \mathcal{A}_2$, for example, explicitly as the map

$$h^1 \otimes v_1 \otimes h^2 \otimes v_2 \otimes h^3 \mapsto \sum_{i,j} (h_{1i}^1 \otimes v_1 \otimes S(h_{2i}^1)) (h_{1j}^2 \otimes v_2 \otimes S(h_{2j}^2)) h^3 \quad (27)$$

($h^1, h^2, h^3 \in H, v_1, v_2 \in V$). Just as the map i_1 can be expressed using the map i and multiplication in \mathcal{A} (recall (18)), the linear automorphism i_n is the composition of

$$i^{\otimes n} \otimes 1 : \mathcal{A}_n = (H \otimes V)^n \otimes H \rightarrow (H \otimes V \otimes H)^{\otimes n} \otimes H$$

with the multiplication map in \mathcal{A} , as in (27).

We combine the linear automorphisms i_n for $n \geq 0$, where we take i_0 to be the identity map on H , to form the graded linear automorphism

$$\mathcal{J} = \coprod_{n \geq 0} i_n : \mathcal{A} \rightarrow \mathcal{A}, \quad (28)$$

so that

$$\mathcal{J}|_H = i_0 = 1 : H \rightarrow H, \quad (29)$$

the identity map, and

$$\mathcal{J}|_{H \otimes V} = i : H \otimes V \xrightarrow{\sim} H \cdot V, \quad (30)$$

a linear isomorphism and in fact an H -module isomorphism (recall (16)). Moreover,

$$\mathcal{J}|_{T(H \otimes V)} : T(H \otimes V) \rightarrow \mathcal{A}$$

is a linear isomorphism from the subalgebra $T(H \otimes V)$ of \mathcal{A} to the subalgebra of \mathcal{A} generated by $H \cdot V$, and this subalgebra is naturally isomorphic to the tensor algebra $T(H \cdot V)$, so that the second statement in the Theorem is proved:

$$\mathcal{J}|_{T(H \otimes V)} : T(H \otimes V) \xrightarrow{\sim} T(H \cdot V); \quad (31)$$

in fact, this map is the isomorphism of tensor algebras induced by the linear isomorphism (30) of generating vector spaces of the two tensor algebras. Thus the linear automorphism \mathcal{J} restricts to algebra isomorphisms (29) and (31), which canonically extends (30), and the linear decomposition

$$\mathcal{A} = T(H \otimes V) \otimes H$$

of the domain of \mathcal{J} transports to a linear decomposition

$$\mathcal{A} = T(H \cdot V) \otimes H \quad (32)$$

of the codomain; the associated canonical linear map

$$T(H \cdot V) \otimes H \rightarrow \mathcal{A} \quad (33)$$

is the map induced by multiplication in \mathcal{A} . In particular, the natural linear map (33) induced by multiplication is a linear isomorphism. This proves the first part of the third statement in the Theorem.

Now we want to describe explicitly the multiplication operation in the algebra \mathcal{A} in terms of the linear tensor product decomposition (32) of \mathcal{A} into the two specified subalgebras of \mathcal{A} . But by the comments before (12), to prove the rest all we need to show is that $T(H \cdot V)$ is stable under the adjoint action of H . But this is clear from the iteration of (8) for a product of several elements of $H \cdot V$, together with (9). This completes the proof of Theorem 2.1. \square

Remark 3.1 *The proof in [1] proceeds by viewing $T(H \otimes V) \# H$ as a universal object and by constructing mutually inverse canonical maps between this structure and \mathcal{A} . The argument above essentially carries this out “concretely.”*

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